Oppgave 1

A) Skriver om til kartesisk koordinatsystem

$$f(s,\theta) = s e_\theta$$

$$= s\left(-e_1 \sin\theta + e_2 \cos\theta\right)$$

$$= -e_1\, s \cdot \sin\theta + e_2\, s \cdot \cos\theta$$

$$= -e_1 \cdot x_2 + e_2 \cdot x_1$$

$$= \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

$$\nabla \times f = \begin{matrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ & \times & & \times & & \\ -x_2 & x_1 & 0 & -x_2 & x_1 & 0 \end{matrix}$$

$$= (0, 0, 2)^T$$

~~Rotasjon~~ : $\cancel{\nabla \times f =}$

~~"Rotation"~~

B]

$$f(s,\theta) = e_\theta$$

$$= -\sin\theta\, e_1 + \cos\theta\, e_2$$

$$= \frac{1}{s}\left(-s\cdot\sin\theta e_1 + s\cdot\cos\theta e_2\right)$$

$$= \frac{-x_2}{\sqrt{x_1^2+x_2^2}}\, e_1 + \frac{x_1}{\sqrt{x_1^2+x_2^2}}\, e_2$$

$$= \begin{pmatrix} \frac{-x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ 0 \end{pmatrix}$$

$$\nabla \times f = \begin{array}{cccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ & \times & \times & \times & & \\ \frac{-x_2}{\sqrt{x_1^2+x_2^2}} & \frac{x_1}{\sqrt{x_1^2+x_2^2}} & 0 & \frac{-x_2}{\sqrt{x_1^2+x_2^2}} & \frac{x_1}{\sqrt{x_1^2+x_2^2}} & 0 \end{array}$$

$$= \left(0, 0, \frac{\sqrt{x_1^2+x_2^2} - \frac{x_1^2}{\sqrt{x_1^2+x_2^2}}}{x_1^2+x_2^2} + \frac{\sqrt{x_1^2+x_2^2} - \frac{x_2^2}{\sqrt{x_1^2+x_2^2}}}{x_1^2+x_2^2}\right)$$

$$= \left(0, 0, \frac{x_1^2 + x_2^2 - x_1^2}{(x_1^2 + x_2^2)^{3/2}} + \frac{x_1^2 + x_2^2 - x_2^2}{(x_1^2 + x_2^2)^{3/2}}\right)$$

$$= \left(0, 0, \frac{x_1^2 + x_2^2}{(x_1^2 + x_2^2)^{3/2}}\right)$$

$$= \left(0, 0, \frac{1}{\sqrt{x_1^2 + x_2^2}}\right) = \underline{\left(0, 0, \frac{1}{s}\right)}$$

$\underline{c}$ $f(s, \theta) = e_\theta / s$

$$= \frac{1}{s^2} \cdot s e_\theta$$

$$= \frac{1}{s^2}\left(-x_2 e_1 + x_1 e_2\right)$$

$$= \frac{1}{x_1^2 + x_2^2}\left(-x_2 e_1 + x_1 e_2\right)$$

$$= \begin{pmatrix} \frac{-x_2}{x_1^2 + x_2^2} \\ \frac{x_1}{x_1^2 + x_2^2} \\ 0 \end{pmatrix}$$

$$\nabla \times f = \begin{array}{cccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} & 0 & \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} & 0 \end{array}$$

$$= \left(0,\ 0,\ \frac{(x_1^2+x_2^2) - x_1 \cdot 2x_1}{(x_1^2+x_2^2)^2} + \frac{(x_1^2+x_2^2) - x_2 \cdot 2x_2}{(x_1^2+x_2^2)^2}\right)$$

$$= \underline{(0,0,0)}$$

Dette gir mening ettersom vektorfeltet er konservativt med potensialet

$$\phi = \arctan\left(\frac{y}{x}\right)$$

D]

$$f(s,\theta) = \frac{e_\theta}{s^2}$$

$$= \frac{1}{s^3} \cdot s e_\theta$$

$$= \frac{1}{(x_1^2+x_2^2)^{3/2}} \cdot \left(-x_2 e_1 + x_1 e_2\right)$$

$$= \begin{pmatrix} \frac{-x_2}{(x_1^2+x_2^2)^{3/2}} \\ \frac{x_1}{(x_1^2+x_2^2)^{3/2}} \\ 0 \end{pmatrix}$$

$$\nabla \times f = \begin{array}{cccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ & & \times & \times & & \\ \frac{-x_2}{(x_1^2+x_2^2)^{3/2}} & \frac{x_1}{(x_1^2+x_2^2)^{3/2}} & 0 & \frac{-x_2}{(x_1^2+x_2^2)^{3/2}} & \frac{x_1}{(x_1^2+x_2^2)^{3/2}} & 0 \end{array}$$

$$= \left(0, 0, \frac{(x_1^2+x_2^2)^{3/2} - x_1 \cdot \frac{3}{2} \cdot 2x_1 \cdot (x_1^2+x_2^2)^{1/2}}{(x_1^2+x_2^2)^3}\right.$$

$$+ \frac{(x_1^2+x_2^2)^{3/2} - x_2 \cdot \frac{3}{2} \cdot 2x_2 \cdot (x_1^2+x_2^2)^{1/2}}{(x_1^2+x_2^2)^3} \Bigg)$$

$$= \left(0, 0, \frac{2(x_1^2+x_2^2)^2 - 3(x_1^2+x_2^2)^2}{(x_1^2+x_2^2)^{7/2}}\right)$$

$$= \left(0, 0, -\frac{1}{(x_1^2+x_2^2)^{3/2}}\right) = \underline{\left(0, 0, -\frac{1}{s^3}\right)}$$

Oppgave 2

Vi gjør oppgave 1, bare i sylinder-koordinater

$$\nabla\times f = \left(\frac{1}{s}\frac{\partial f_z}{\partial\theta} - \frac{\partial f_\theta}{\partial z}\right)e_s + \left(\frac{\partial f_s}{\partial z} - \frac{\partial f_z}{\partial s}\right)e_\theta$$

$$+ \frac{1}{s}\left(\frac{\partial(s f_\theta)}{\partial s} - \frac{\partial f_s}{\partial\theta}\right)e_z$$

A] $f(s,\theta) = s e_\theta = (0, s, 0)$

$$\nabla\times f = \left(\frac{1}{s}\cdot 0 - 0\right)e_s + (0-0)e_\theta$$

$$+ \frac{1}{s}(2s - 0)e_z = \underline{(0, 0, 2)}$$

B] $f(s,\theta) = e_\theta = (0, 1, 0)$

$$\nabla\times f = \left(\frac{1}{s}\cdot 0 - 0\right)e_s + (0-0)e_\theta$$

$$+ \frac{1}{s}(1-0)e_z = \left(0, 0, \frac{1}{s}\right)$$

C] $f(s,\theta) = e_\theta/s = (0, 1/s, 0)$

$$\nabla \times f = \left(\frac{1}{s}\cdot 0 - 0\right)e_s + (0-0)\,e_\theta$$

$$+ \frac{1}{s}(0-0)\,e_z$$

$$= \underline{(0,0,0)}$$

For potensialet:

Gradienten i sylinderkoordinater er

$$\nabla f = \frac{\partial f}{\partial s}e_s + \frac{1}{s}\frac{\partial f}{\partial \theta}e_\theta + \frac{\partial f}{\partial z}e_z$$

og vi kan se f har potensialet

$$\underline{\phi = \theta}$$

D] $f(s,\theta) = \frac{e_\theta}{s^2} = (0, 1/s^2, 0)$

$$\nabla \times f = \left(\frac{1}{s}\cdot 0 - 0\right)e_s + (0-0)\,e_\theta$$

$$+ \frac{1}{s}\left(-\frac{1}{s^2} - 0\right)e_z = \left(0,0,-\frac{1}{s^3}\right)$$

Oppgave 3

Tips

Hvis F har et vektorpotensiale G kan vi anta at $G = (G_1, G_2, 0)$.

Bevis

La H være et vektorpotensiale for F. $H = (H_1, H_2, H_3)$. Velg et skalarfelt slik at $\frac{\partial \Phi}{\partial z} = -H_3$, og la $G = H + \nabla \Phi$.

Vi får $G_3 = H_3 + \frac{\partial \Phi}{\partial z} = 0$, og

$$\nabla \times G = \nabla \times (H + \nabla \Phi) = \nabla H + \nabla \times (\nabla \Phi) = \nabla H$$

Eksemplene ble regnet ut i forelesning, er samme utregning som er gjort mange ganger før.