Oppgave 1

Disse to metodene er ikke egentlig forskjellige, men den andre metoden kan være en enklere måte å huske hvordan det skal gjøres på.

Vi starter med en standard teknikk, determinant:

$$\det\left(\begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}\right)$$

$$= i(2\cdot 6 - 3\cdot 5) - j(1\cdot 6 - 3\cdot 4) + k(1\cdot 5 - 2\cdot 4)$$

$$= i(-3) + j(6) + k(-3)$$

$$= \underline{(-3, 6, -3)}$$

Neste setter vi det opp slik

$$\begin{matrix}1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6\end{matrix}$$

og tar multiplikasjonene/subtraksjonene gitt av 2×2-matriser bortover (determinant)

$$\begin{matrix}1 & 2 & 3 & 1 & 2 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6\end{matrix}$$

Kryssproduktet blir

$$(2\cdot 6 - 3\cdot 5,\ 3\cdot 4 - 1\cdot 6,\ 1\cdot 5 - 2\cdot 4)$$

$$= \underline{(-3,\ 6,\ -3)}$$

Generelt kan vi bruke denne for å ta vektorprodukt av to generelle vektorer

$$\begin{array}{cccccc} a & b & c & a & b & c \\ d & e & f & d & e & f \end{array}$$

som gir

$$(a, b, c) \times (d, e, f)$$

$$= \left(\det\begin{pmatrix} b & c \\ e & f \end{pmatrix}, \det\begin{pmatrix} c & a \\ f & d \end{pmatrix}, \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} \right)$$

$$= (bf - ce, cd - af, ae - bd)$$

som vi er kjent med fra før.

Oppgave 2

"Triks": La $f = (x, y, z)$ og se på

$$i = \frac{\partial f}{\partial x}, \quad j = \frac{\partial f}{\partial y}, \quad k = \frac{\partial f}{\partial z}.$$

Starter med sylinder:

Vi har
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Vi vil finne $\hat{r} = \frac{\partial f}{\partial r}$, $\hat{\theta} = \frac{\partial f}{\partial \theta}$ og $\hat{z} = \frac{\partial f}{\partial z}$.

(og normaliserer de for enhetsvektorer)

$$\hat{r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} = \cos\theta \cdot i + \sin\theta \cdot j$$
$$= \underline{(\cos\theta, \sin\theta, 0)}$$

normalisert

$$\hat{\theta} = \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -\sin\theta\, i + \cos\theta \cdot j = \underline{(-\sin\theta, \cos\theta, 0)}$$

$$\hat{z} = k = \underline{(0, 0, 1)}$$

Kule:

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

vi vil ha

$$\hat{r} = \frac{\partial f}{\partial r}, \quad \hat{\theta} = \frac{\partial f}{\partial \theta}, \quad \hat{\varphi} = \frac{\partial f}{\partial \varphi}$$

$$\hat{r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= \sin\theta\cos\varphi\, i + \sin\theta\sin\varphi\, j + \cos\theta\, k$$

$$= \underline{(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)}$$

normalisering

$$\hat{\theta} = \frac{\partial f}{\partial \theta} = (\text{kjerneregel}) = \cos\theta\cos\varphi\, i + \cos\theta\sin\varphi\, j - \sin\theta\, k$$

$$= \cancel{(\cos\theta\cos\varphi, \cos\theta}$$

$$= \underline{(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)}$$

normalisering

$$\hat{\varphi} = \frac{\partial f}{\partial \varphi} = (\text{kjerneregel}) = -\sin\varphi\, \hat{i} + \cos\varphi\, \hat{j}$$

$$= \underline{(-\sin\varphi, \cos\varphi, 0)}$$

Oppgave 3

1. $(x_1, x_2, x_3)^T$

$$\nabla f = \begin{pmatrix} \nabla(x_1) \\ \nabla(x_2) \\ \nabla(x_3) \end{pmatrix} = \underline{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\nabla \cdot f = \frac{\partial}{\partial x_1}(x_1) + \frac{\partial}{\partial x_2}(x_2) + \frac{\partial}{\partial x_3}(x_3)$$

$$= 1 + 1 + 1 = \underline{3}$$

$$\nabla \times f = \begin{matrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ & \times & \times & \times & & \\ x_1 & x_2 & x_3 & x_1 & x_2 & x_3 \end{matrix}$$

$$= \left(\frac{\partial}{\partial x_2}(x_3) - \frac{\partial}{\partial x_3}(x_2), \frac{\partial}{\partial x_3}(x_1) - \frac{\partial}{\partial x_1}(x_3), \frac{\partial}{\partial x_1}(x_2) - \frac{\partial}{\partial x_2}(x_1) \right) = \underline{(0, 0, 0)}$$

Dette kan vi også se fordi

$$f = \nabla \phi \quad \text{der} \quad \phi = \frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2}$$

og konservative felt er sirkulasjonsfrie.

2. $(x_3, 0, 0)^T$

$$\nabla f = \begin{pmatrix} \nabla (x_3) \\ \nabla (0) \\ \nabla (0) \end{pmatrix} = \underline{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\nabla \cdot f = \frac{\partial}{\partial x_1}(x_3) = \underline{0}$$

$$\nabla \times f = \begin{array}{cccccccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \times & \frac{\partial}{\partial x_3} & \times & \frac{\partial}{\partial x_1} & \times & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ x_3 & 0 & & 0 & & x_3 & & 0 & 0 \end{array}$$

$$= \underline{(0, 1, 0)}$$

3. $(-x_2, x_1, 0)^T$

$$\nabla f = \begin{pmatrix} \nabla (-x_2) \\ \nabla (x_1) \\ \nabla (0) \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\nabla \cdot f = \frac{\partial}{\partial x_1}(-x_2) + \frac{\partial}{\partial x_2}(x_1) + \frac{\partial}{\partial x_3}(0) = \underline{0}$$

$$\nabla \times f = \begin{array}{cccccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \times & \frac{\partial}{\partial x_3} & \times & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ -x_2 & x_1 & & 0 & & -x_2 & x_1 & 0 \end{array} = \underline{(0, 0, 2)}$$

Oppgave 4

$a \times (b \times c)$ er en vektor i planet utspent av b og c, imens $(a \times b) \times c$ er en vektor i planet utspent av a og b, så det burde ikke generelt bli samme vektor.

Eksempel på der de er ikke like

$a = i$

$b = i + j$

$c = k$

$a \times (b \times c) = -k$

$(a \times b) \times c = 0$

<u>Utregningene:</u> $a \times (b \times c) = i((i+j) \times k) = i \times (-j + i) = \underline{-k}$

$(a \times b) \times c = (i \times (i+j)) \times k = (k) \times k = \underline{0}$

Oppgave 5

$f : \mathbb{R}^3 \to \mathbb{R}, \quad g : \mathbb{R}^3 \to \mathbb{R}^3$

1. $$\nabla \circ (\nabla f) = \nabla \circ \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

Dette er operatoren Δ (Laplace-operatoren) som dere allerede er kjent med,

$$\Delta f = \nabla \circ (\nabla f)$$

2. $$\nabla \times (\nabla f) = \nabla \times \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

$$= \begin{array}{cccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{array}$$

$$= \left(\frac{\partial^2 f}{\partial x_2 \partial x_3} - \frac{\partial^2 f}{\partial x_3 \partial x_2}, \frac{\partial^2 f}{\partial x_3 \partial x_1} - \frac{\partial^2 f}{\partial x_1 \partial x_3}, \frac{\partial^2 f}{\partial x_1 \partial x_2} - \frac{\partial^2 f}{\partial x_2 \partial x_1} \right)$$

I de fleste tilfeller har der ivasjons-rekkefølgen ikke noe å si, og om dette er tilfellet:

$$= (0, 0, 0)$$

Kan igjen finne det direkte ved at konservative felt er sirkulasjonsfrie

3. $\nabla(\nabla \cdot g) = \nabla\left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}\right)$

$$= \left(\frac{\partial^2 g_1}{\partial x_1^2} + \frac{\partial^2 g_2}{\partial x_1 \partial x_2} + \frac{\partial^2 g_3}{\partial x_1 \partial x_3},\right.$$

$$\frac{\partial^2 g_1}{\partial x_2 \partial x_1} \quad \frac{\partial^2 g_2}{\partial x_2^2} \quad \frac{\partial^2 g_3}{\partial x_2 \partial x_3},$$

$$\left.\frac{\partial^2 g_1}{\partial x_3 \partial x_1} \quad \frac{\partial^2 g_2}{\partial x_3 \partial x_2} \quad \frac{\partial^2 g_3}{\partial x_3^2}\right)$$

4. $\nabla \cdot (\nabla \times g) = \nabla \cdot \left(\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3}, \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1}, \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2}\right)$

$$= \frac{\partial^2 g_3}{\partial x_1 \partial x_2} - \frac{\partial^2 g_2}{\partial x_1 \partial x_3} + \frac{\partial^2 g_1}{\partial x_2 \partial x_3} - \frac{\partial^2 g_3}{\partial x_2 \partial x_1} + \frac{\partial^2 g_2}{\partial x_3 \partial x_1} - \frac{\partial^2 g_1}{\partial x_1 \partial x_2}$$

Dessom derivasjonsrekkefølgen ikke har noe å si:

$$= \underline{0}$$

5. $\nabla \times (\nabla \times g) = \nabla \times \left(\frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3} , \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1} , \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} \right)$

$$= \begin{array}{cccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3} & \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1} & \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} & \frac{\partial g_3}{\partial x_2} - \frac{\partial g_2}{\partial x_3} & \frac{\partial g_1}{\partial x_3} - \frac{\partial g_3}{\partial x_1} & \frac{\partial g_2}{\partial x_1} - \frac{\partial g_1}{\partial x_2} \end{array}$$

$$= \left(\frac{\partial^2 g_2}{\partial x_2 \partial x_1} - \frac{\partial^2 g_1}{\partial x_2^2} - \frac{\partial^2 g_1}{\partial x_3^2} + \frac{\partial^2 g_3}{\partial x_3 \partial x_1} \right.,$$

$$\frac{\partial^2 g_3}{\partial x_3 \partial x_2} - \frac{\partial^2 g_2}{\partial x_3^2} - \frac{\partial^2 g_2}{\partial x_1^2} + \frac{\partial^2 g_1}{\partial x_1 \partial x_2},$$

$$\left. \frac{\partial^2 g_1}{\partial x_1 \partial x_3} - \frac{\partial^2 g_3}{\partial x_1^2} - \frac{\partial^2 g_3}{\partial x_2^2} + \frac{\partial^2 g_2}{\partial x_2 \partial x_3} \right)$$